A Temporal Theory for the Basic Formal Ontology: Theorem Proofs

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Abstract. This document contains theorem proofs for the paper *A Temporal Theory for the Basic Formal Ontology*.

Section 1: Introduction

In the following we use P, P', $P_1, \ldots$ and p, p', $p_1, \ldots$ to range over occurrent classes and instances, respectively. We use C, C', $C_1, \ldots$ and c, c', $c_1, \ldots$ to range over continuant classes and instances. We also use U, U', $U_1, \ldots$ and u, u', $u_1, \ldots$ to range over spatiotemporal regional classes and instances, r, r', $r_1, \ldots$ to range over spatial regions and t, t', $t_1, \ldots$ to range over temporal regions. Relations between classes are depicted in $italics$, whereas all other relations are depicted in **bold**. The logical connectors $\neg$, $=$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$ have their usual interpretation. The symbol $=_{def}$ is used for definitions, $\forall$ for universal quantification, $\exists$ for existential quantification, and $\exists! a$ abbreviates a statement to the effect that a unique object a exists. We usually omit leading universal quantifiers in our formulae. Names of axioms begin with 'A', names of definitions begin with 'D', names of lemmata begin with 'L', and names of theorems begin with 'T'.

$$P_1 \; is\_a \; P_2 =_{def} \forall p. \;\; p \; \textbf{instance\_of} \; P_1 \Rightarrow p \; \textbf{instance\_of} \; P_2 \tag{D1.1}$$

$$C_1 \; is\_a \; C_2 =_{def} \forall c, t. \;\; c \; \textbf{instance\_of} \; C_1 \; \textbf{at} \; t \Rightarrow c \; \textbf{instance\_of} \; C_2 \; \textbf{at} \; t \tag{D1.2}$$

$$P \; is\_a \; P \tag{L1.1}$$

$$C \; is\_a \; C \tag{L1.2}$$

$$p \; \textbf{instance\_of} \; P_1 \wedge P_1 \; is\_a \; P_2 \Rightarrow p \; \textbf{instance\_of} \; P_2 \tag{L1.3}$$

$$c \; \textbf{instance\_of} \; C_1 \; \textbf{at} \; t \wedge C_1 \; is\_a \; C_2 \Rightarrow c \; \textbf{instance\_of} \; C_2 \; \textbf{at} \; t \tag{L1.4}$$

$$P_1 \; is\_a \; P_2 \wedge P_2 \; is\_a \; P_3 \Rightarrow P_1 \; is\_a \; P_3 \tag{L1.5}$$

$$C_1 \; is\_a \; C_2 \wedge C_2 \; is\_a \; C_3 \Rightarrow C_1 \; is\_a \; C_3 \tag{L1.6}$$

Proof. L1.1 and L1.2 follow from D1.1 and D1.2 respectively. L1.3 and L1.4 can be proven by *modus ponens* with D1.1 and D1.2 respectively. L1.5 and L1.6 can be proven by the transitivity of implication with D1.1 and D1.2 respectively. □

$$p_1 \; \textbf{identical\_to} \; p_2 \Rightarrow (\forall P. \;\; p_1 \; \textbf{instance\_of} \; P \Leftrightarrow p_2 \; \textbf{instance\_of} \; P) \tag{A1.1}$$

$$c_1 \; \textbf{identical\_to} \; c_2 \; \textbf{at} \; t \Rightarrow (\forall C. \;\; c_1 \; \textbf{instance\_of} \; C \; \textbf{at} \; t \Leftrightarrow c_2 \; \textbf{instance\_of} \; C \; \textbf{at} \; t) \tag{A1.2}$$

Henceforth we write $p : P$ as an abbreviation for p **instance_of** P and $c : C$ **at** t as an abbreviation for c **instance_of** C **at** t. Furthermore we write $p_1, \ldots, p_n : P$ as an abbreviation of $p_1 : P \wedge \ldots \wedge p_n : P$ and $c_1, \ldots, c_n : C$ **at** t as an abbreviation of $c_1 : C$ **at** $t \wedge \ldots \wedge c_n : C$ **at** t.

$$p_1 \textbf{ identical\_to } p_2 \Rightarrow p_2 \textbf{ identical\_to } p_1 \tag{L1.7}$$

$$c_1 \textbf{ identical\_to } c_2 \textbf{ at } t \Rightarrow c_2 \textbf{ identical\_to } c_1 \textbf{ at } t \tag{L1.8}$$

$$p_1 : P \wedge p_1 \textbf{ identical\_to } p_2 \Rightarrow p_2 : P \tag{L1.9}$$

$$c_1 : C \textbf{ at } t \wedge c_1 \textbf{ identical\_to } c_2 \textbf{ at } t \Rightarrow c_2 : C \textbf{ at } t \tag{L1.10}$$

Proof. L1.7 and L1.8 can be proved by A1.1 and A1.2 respectively. L1.9 and L1.10 can be proven by *modus ponens* with A1.1 and A1.2 respectively. □

$$(U = U_1 \cup U_2) \wedge (U_1 \cap U_2 = \emptyset) \wedge u : U \Rightarrow u : U_1 \vee u : U_2 \tag{A1.3}$$

$$(U = U_1 \cup U_2) \wedge (U_1 \cap U_2 = \emptyset) \wedge U \textit{ is\_a } U' \Rightarrow U_1 \textit{ is\_a } U' \wedge U_2 \textit{ is\_a } U' \tag{A1.4}$$

$$p_1 \textbf{ overlaps } p_2 =_{def} \exists p.\ \ p \textbf{ part\_of } p_1 \wedge p \textbf{ part\_of } p_2 \tag{D1.3}$$

$$c_1 \textbf{ overlaps } c_2 \textbf{ at } t =_{def} \exists c.\ \ c \textbf{ part\_of } c_1 \textbf{ at } t \wedge c \textbf{ part\_of } c_2 \textbf{ at } t \tag{D1.4}$$

$$p_1 \textbf{ discrete\_from } p_2 =_{def} \neg(p_1 \textbf{ overlaps } p_2) \tag{D1.5}$$

$$c_1 \textbf{ discrete\_from } c_2 \textbf{ at } t =_{def} \neg(c_1 \textbf{ overlaps } c_2 \textbf{ at } t) \tag{D1.6}$$

$$p_1 \textbf{ overlaps } p_2 \Leftrightarrow p_2 \textbf{ overlaps } p_1 \tag{L1.11}$$

$$c_1 \textbf{ overlaps } c_2 \textbf{ at } t \Leftrightarrow c_2 \textbf{ overlaps } c_1 \textbf{ at } t \tag{L1.12}$$

Proof. L1.11 and L1.12 can be proved by unfolding D1.3 and D1.4. □

$$p_1 \textbf{ part\_of } p_2 \Leftrightarrow (\forall p.\ \ p \textbf{ overlaps } p_1 \Rightarrow p \textbf{ overlaps } p_2) \tag{A1.5}$$

$$c_1 \textbf{ part\_of } c_2 \textbf{ at } t \Leftrightarrow (\forall c.\ \ c \textbf{ overlaps } c_1 \textbf{ at } t \Rightarrow c \textbf{ overlaps } c_2 \textbf{ at } t) \tag{A1.6}$$

$$p_1 \textbf{ part\_of } p_2 \wedge p_2 \textbf{ part\_of } p_1 \Leftrightarrow p_1 \textbf{ identical\_to } p_2 \tag{A1.7}$$

$$c_1 \textbf{ part\_of } c_2 \textbf{ at } t \wedge c_2 \textbf{ part\_of } c_1 \textbf{ at } t \Leftrightarrow c_1 \textbf{ identical\_to } c_2 \textbf{ at } t \tag{A1.8}$$

$$p_1 \textbf{ part\_of } p_2 \wedge p_2 \textbf{ part\_of } p_3 \Rightarrow p_1 \textbf{ part\_of } p_3 \tag{L1.13}$$

$$c_1 \textbf{ part\_of } c_2 \textbf{ at } t \wedge c_2 \textbf{ part\_of } c_3 \textbf{ at } t \Rightarrow c_1 \textbf{ part\_of } c_3 \textbf{ at } t \tag{L1.14}$$

$$p \textbf{ part\_of } p \tag{L1.15}$$

$$c \textbf{ part\_of } c \textbf{ at } t \tag{L1.16}$$

Proof. L1.13 and L1.15 can be proved by A1.5, whereas L1.14 and L1.16 can be proved by A1.6. □

$$p_1 \textbf{ part\_of } p_2 \wedge p_2 \textbf{ identical\_to } p_3 \Rightarrow p_1 \textbf{ part\_of } p_3 \tag{L1.17}$$

$$p_1 \textbf{ part\_of } p_3 \wedge p_1 \textbf{ identical\_to } p_2 \Rightarrow p_2 \textbf{ part\_of } p_3 \tag{L1.18}$$

$$c_1 \textbf{ part\_of } c_2 \textbf{ at } t \wedge c_2 \textbf{ identical\_to } c_3 \textbf{ at } t \Rightarrow c_1 \textbf{ part\_of } c_3 \textbf{ at } t \tag{L1.19}$$

$$c_1 \textbf{ part\_of } c_3 \textbf{ at } t \wedge c_1 \textbf{ identical\_to } c_2 \textbf{ at } t \Rightarrow c_2 \textbf{ part\_of } c_3 \textbf{ at } t \tag{L1.20}$$

Proof. L1.17 and L1.18 can be proved by A1.7 and L1.13, whereas L1.19 and L1.20 can be proved by A1.8 and L1.14. □

$$P_1 \; \mathit{part\_of} \; P_2 =_{def} \forall p_1.\; p_1 : P_1 \Rightarrow \exists p_2.\; p_2 : P_2 \wedge p_1 \;\textbf{part\_of}\; p_2 \quad \text{(D1.7)}$$

$$C_1 \; \mathit{part\_of} \; C_2 =_{def} \forall c_1, t.\; c_1 : C_1 \;\textbf{at}\; t \Rightarrow \exists c_2.\; c_2 : C_2 \;\textbf{at}\; t \wedge c_1 \;\textbf{part\_of}\; c_2 \;\textbf{at}\; t \quad \text{(D1.8)}$$

$$p_1 \;\textbf{proper\_part\_of}\; p_2 =_{def} p_1 \;\textbf{part\_of}\; p_2 \wedge \neg(p_1 \;\textbf{identical\_to}\; p_2) \quad \text{(D1.9)}$$

$$c_1 \;\textbf{proper\_part\_of}\; c_2 \;\textbf{at}\; t =_{def} c_1 \;\textbf{part\_of}\; c_2 \;\textbf{at}\; t \wedge \neg(c_1 \;\textbf{identical\_to}\; c_2 \;\textbf{at}\; t) \quad \text{(D1.10)}$$

$$p_1 \;\textbf{partially\_overlaps}\; p_2 =_{def} p_1 \;\textbf{overlaps}\; p_2 \wedge \neg(p_1 \;\textbf{part\_of}\; p_2) \wedge \neg(p_2 \;\textbf{part\_of}\; p_1) \quad \text{(D1.11)}$$

$$c_1 \;\textbf{partially\_overlaps}\; c_2 \;\textbf{at}\; t =_{def} c_1 \;\textbf{overlaps}\; c_2 \;\textbf{at}\; t \wedge \neg(c_1 \;\textbf{part\_of}\; c_2 \;\textbf{at}\; t) \wedge \neg(c_2 \;\textbf{part\_of}\; c_1 \;\textbf{at}\; t) \quad \text{(D1.12)}$$

$$p_1 \;\textbf{overlaps}\; p_2 \Rightarrow p_1 \;\textbf{partially\_overlaps}\; p_2 \vee p_1 \;\textbf{proper\_part\_of}\; p_2 \vee p_2 \;\textbf{proper\_part\_of}\; p_1 \vee p_1 \;\textbf{identical\_to}\; p_2 \quad \text{(L1.21)}$$

$$c_1 \;\textbf{overlaps}\; c_2 \;\textbf{at}\; t \Rightarrow c_1 \;\textbf{partially\_overlaps}\; c_2 \;\textbf{at}\; t \vee c_1 \;\textbf{proper\_part\_of}\; c_2 \;\textbf{at}\; t \vee c_2 \;\textbf{proper\_part\_of}\; c_1 \;\textbf{at}\; t \vee c_1 \;\textbf{identical\_to}\; c_2 \;\textbf{at}\; t \quad \text{(L1.22)}$$

Proof. In order to prove L1.21, consider the case where $\neg(p_1$ **identical_to** $p_2)$. By D1.3, there is some p such that p **part_of** p_1 and p **part_of** p_2. For the cases where p_1 **part_of** p_2 and p_2 **part_of** p_1, then p_1 **proper_part_of** p_2 and p_2 **proper_part_of** p_1 by D1.9. If $\neg(p_1$ **part_of** $p_2)$ and $\neg(p_2$ **part_of** $p_1)$ then p_1 **partially_overlaps** p_2 by D1.11. We construct a similar proof for L1.22. □

Section 2: Connected Regions

$$\mathit{Spatiotemporal\_Region} \; \mathit{is\_a} \; \mathit{Occurrent} \quad \text{(A2.1)}$$

$$\mathit{Temporal\_Region} \; \mathit{is\_a} \; \mathit{Occurrent} \quad \text{(A2.2)}$$

$$\mathit{Connected\_Spatiotemporal\_Region} \; \mathit{is\_a} \; \mathit{Spatiotemporal\_Region} \quad \text{(A2.3)}$$

$$\mathit{Connected\_Temporal\_Region} \; \mathit{is\_a} \; \mathit{Temporal\_Region} \quad \text{(A2.4)}$$

$$\mathit{Spatial\_Region} \; \mathit{is\_a} \; \mathit{Continuant} \quad \text{(A2.5)}$$

$$\mathit{Connected\_Spatial\_Region} \; \mathit{is\_a} \; \mathit{Spatial\_Region} \quad \text{(A2.6)}$$

$$(\mathit{Connected\_Spatiotemporal\_Region} = \mathit{Connected\_Spatiotemporal\_Instant} \cup \mathit{Connected\_Spatiotemporal\_Interval}) \wedge (\mathit{Connected\_Spatiotemporal\_Instant} \cap \mathit{Connected\_Spatiotemporal\_Interval} = \emptyset) \quad \text{(A2.7)}$$

$$\mathit{Connected\_Spatiotemporal\_Instant} \; \mathit{is\_a} \; \mathit{Connected\_Spatiotemporal\_Region} \quad \text{(L2.1)}$$

$$\mathit{Connected\_Spatiotemporal\_Interval} \; \mathit{is\_a} \; \mathit{Connected\_Spatiotemporal\_Region} \quad \text{(L2.2)}$$

$$u : \mathit{Connected\_Spatiotemporal\_Region} \Rightarrow u : \mathit{Connected\_Spatiotemporal\_Instant} \vee u : \mathit{Connected\_Spatiotemporal\_Interval} \quad \text{(T2.1)}$$

Proof. L2.1 and L2.2 can be proved by A2.7, A1.4 and L1.1, whereas T2.1 can be proved by A2.7 and A1.3. □

$$u_1 : \mathit{Connected\_Spatiotemporal\_Region} \wedge u_2 : \mathit{Connected\_Spatiotemporal\_Instant} \wedge u_1 \ \textbf{part\_of}\ u_2 \Rightarrow u_1 : \mathit{Connected\_Spatiotemporal\_Instant} \quad \text{(A2.8)}$$

$$u_1 : \mathit{Connected\_Spatiotemporal\_Interval} \wedge u_2 : \mathit{Connected\_Spatiotemporal\_Region} \wedge u_1 \ \textbf{part\_of}\ u_2 \Rightarrow u_2 : \mathit{Connected\_Spatiotemporal\_Interval} \quad \text{(A2.9)}$$

$$\exists! \mathcal{U}.\ \mathcal{U} : \mathit{Connected\_Spatiotemporal\_Interval} \quad \text{(A2.10)}$$

$$u : \mathit{Connected\_Spatiotemporal\_Region} \Rightarrow u \ \textbf{part\_of}\ \mathcal{U} \quad \text{(A2.11)}$$

$$\mathcal{U} : \mathit{Connected\_Spatiotemporal\_Region} \quad \text{(T2.2)}$$

$$\mathcal{U} : \mathit{Spatiotemporal\_Region} \quad \text{(L2.3)}$$

$$u : \mathit{Connected\_Spatiotemporal\_Region} \wedge \mathcal{U} \ \textbf{part\_of}\ u \Rightarrow u \ \textbf{identical\_to}\ \mathcal{U} \quad \text{(T2.3)}$$

Proof. T2.2 can be proved by A2.10, L2.2 and L1.3. L2.3 can be proved by T2.2, A2.3 and L1.3, whereas T2.3 can be proved by A2.11 and A1.7. □

$$u_1, u_2 : \mathit{Connected\_Spatiotemporal\_Region} \wedge u_1 \ \textbf{part\_of}\ u_2 \Rightarrow \mathit{time}(u_1) \ \textbf{part\_of}\ \mathit{time}(u_2) \quad \text{(A2.12)}$$

$$u : \mathit{Connected\_Spatiotemporal\_Region} \Rightarrow \mathit{time}(\mathit{time}(u)) \ \textbf{identical\_to}\ \mathit{time}(u) \quad \text{(A2.13)}$$

$$\mathcal{T} =_{def} \mathit{time}(\mathcal{U}) \quad \text{(D2.1)}$$

$$\mathcal{T} \ \textbf{identical\_to}\ \mathit{time}(\mathcal{T}) \quad \text{(T2.4)}$$

$$u : \mathit{Connected\_Spatiotemporal\_Region} \Rightarrow \mathit{time}(u) \ \textbf{part\_of}\ \mathcal{T} \quad \text{(T2.5)}$$

$$u_1, u_2 : \mathit{Connected\_Spatiotemporal\_Region} \wedge u_1 \ \textbf{identical\_to}\ u_2 \Rightarrow \mathit{time}(u_1) \ \textbf{identical\_to}\ \mathit{time}(u_2) \quad \text{(T2.6)}$$

Proof. T2.4 can be proved by A2.13, T2.2, L1.7 and D2.1. T2.5 can be proved by A2.12, A2.11 and D2.1. T2.6 can be proved by A2.12 and A1.7. □

$$u : \mathit{Connected\_Spatiotemporal\_Region} \Rightarrow \exists t.\ \mathit{time}(u) \ \textbf{identical\_to}\ t \wedge t : \mathit{Connected\_Temporal\_Region} \quad \text{(A2.14)}$$

$$t : \mathit{Connected\_Temporal\_Region} \Rightarrow \exists u.\ \mathit{time}(u) \ \textbf{identical\_to}\ t \wedge u : \mathit{Connected\_Spatiotemporal\_Region} \quad \text{(A2.15)}$$

$$\mathcal{T} : \mathit{Connected\_Temporal\_Region} \quad \text{(T2.7)}$$

$$\mathcal{T} : \mathit{Temporal\_Region} \quad \text{(L2.4)}$$

$$u : \mathit{Connected\_Spatiotemporal\_Instant} \Rightarrow \mathit{time}(u) : \mathit{Connected\_Temporal\_Region} \quad \text{(L2.5)}$$

$$u : \mathit{Connected\_Spatiotemporal\_Interval} \Rightarrow \mathit{time}(u) : \mathit{Connected\_Temporal\_Region} \quad \text{(L2.6)}$$

$$t : \mathit{Connected\_Temporal\_Region} \Rightarrow t \ \textbf{part\_of}\ \mathcal{T} \quad \text{(L2.7)}$$

Proof. T2.7 can be proved by T2.2, A2.14 and D2.1, whereas L2.4 can be proved by T2.7, A2.4 and L1.3. L2.5 and L2.6 can be proved by T2.1 and A2.14. L2.7 can be proved by A2.15, T2.5 and L1.18. □

$$t\colon \mathit{Connected\_Temporal\_Instant} =_{def} t\colon \mathit{Connected\_Temporal\_Region} \wedge \mathit{duration}(t) = 0 \quad \text{(D2.2)}$$

$$t\colon \mathit{Connected\_Temporal\_Interval} =_{def} t\colon \mathit{Connected\_Temporal\_Region} \wedge \mathit{duration}(t) > 0 \quad \text{(D2.3)}$$

$$u\colon \mathit{Connected\_Spatiotemporal\_Instant} \Rightarrow \mathit{duration}(\mathit{time}(u)) = 0 \quad \text{(A2.16)}$$

$$u\colon \mathit{Connected\_Spatiotemporal\_Interval} \Rightarrow \mathit{duration}(\mathit{time}(u)) > 0 \quad \text{(A2.17)}$$

$$u\colon \mathit{Connected\_Spatiotemporal\_Instant} \Rightarrow \mathit{time}(u)\colon \mathit{Connected\_Temporal\_Instant} \quad \text{(L2.8)}$$

$$u\colon \mathit{Connected\_Spatiotemporal\_Interval} \Rightarrow \mathit{time}(u)\colon \mathit{Connected\_Temporal\_Interval} \quad \text{(L2.9)}$$

Proof. L2.8 can be proved by A2.16, L2.5 and D2.2, whereas L2.9 can be proved by A2.17, L2.6 and D2.3. □

$$t_1, t_2\colon \mathit{Connected\_Temporal\_Region} \wedge t_1 \ \textbf{part\_of}\ t_2 \Rightarrow \mathit{duration}(t_1) \leq \mathit{duration}(t_2) \quad \text{(A2.18)}$$

$$u_1, u_2\colon \mathit{Connected\_Spatiotemporal\_Region} \wedge u_1 \ \textbf{part\_of}\ u_2 \Rightarrow \mathit{space}(u_1) \ \textbf{part\_of}\ \mathit{space}(u_2) \quad \text{(A2.19)}$$

$$u\colon \mathit{Connected\_Spatiotemporal\_Region} \Rightarrow \mathit{space}(\mathit{space}(u)) \ \textbf{identical\_to}\ \mathit{space}(u) \quad \text{(A2.20)}$$

$$\mathcal{R} =_{def} \mathit{space}(\mathcal{U}) \quad \text{(D2.4)}$$

$$\mathcal{R} \ \textbf{identical\_to}\ \mathit{space}(\mathcal{R}) \quad \text{(T2.8)}$$

$$u\colon \mathit{Connected\_Spatiotemporal\_Region} \Rightarrow \mathit{space}(u) \ \textbf{part\_of}\ \mathcal{R} \quad \text{(T2.9)}$$

$$u_1, u_2\colon \mathit{Connected\_Spatiotemporal\_Region} \wedge u_1 \ \textbf{identical\_to}\ u_2 \Rightarrow \mathit{space}(u_1) \ \textbf{identical\_to}\ \mathit{space}(u_2) \quad \text{(T2.10)}$$

Proof. T2.8 can be proved by A2.20, T2.2, L1.7 and D2.4. T2.9 can be proved by A2.19, A2.11 and D2.4. T2.10 can be proved by A2.19 and A1.7. □

$$u\colon \mathit{Connected\_Spatiotemporal\_Region} \Rightarrow \exists r.\ \ \mathit{space}(u) \ \textbf{identical\_to}\ r \wedge r\colon \mathit{Connected\_Spatial\_Region} \quad \text{(A2.21)}$$

$$r\colon \mathit{Connected\_Spatial\_Region} \Rightarrow \exists u.\ \ \mathit{space}(u) \ \textbf{identical\_to}\ r \wedge u\colon \mathit{Connected\_Spatiotemporal\_Region} \quad \text{(A2.22)}$$

$$\mathcal{R}\colon \mathit{Connected\_Spatial\_Region} \quad \text{(T2.11)}$$

$$\mathcal{R}\colon \mathit{Spatial\_Region} \quad \text{(L2.10)}$$

$$r\colon \mathit{Connected\_Spatial\_Region} \Rightarrow r \ \textbf{part\_of}\ \mathcal{R} \quad \text{(L2.11)}$$

Proof. T2.11 can be proved by T2.2, A2.21 and D2.4. L2.10 can be proved by T2.11, A2.6 and L1.3. L2.11 can be proved by A2.22, T2.9 and L1.18. □

Section 3: A Temporal Theory for Connected Temporal Regions

$$t_1, t_2, t, t' : \mathit{Connected\_Temporal\_Region} \wedge t_1 \textbf{ meets } t \wedge t_1 \textbf{ meets } t' \wedge t_2 \textbf{ meets } t \Rightarrow t_2 \textbf{ meets } t' \quad \text{(A3.1)}$$

$$t_1, t_2, t, t' : \mathit{Connected\_Temporal\_Region} \wedge t_1 \textbf{ meets } t \wedge t \textbf{ meets } t_2 \wedge t_1 \textbf{ meets } t' \wedge t' \textbf{ meets } t_2 \Rightarrow t \textbf{ identical\_to } t' \quad \text{(A3.2)}$$

$$t_1, t_2 : \mathit{Connected\_Temporal\_Region} \wedge t_1 \textbf{ meets } t_2 \Rightarrow (t_1 : \mathit{Connected\_Temporal\_Interval} \vee t_2 : \mathit{Connected\_Temporal\_Interval}) \quad \text{(A3.3)}$$

$$t_1, t_2, t_3, t_4 : \mathit{Connected\_Temporal\_Region} \wedge t_1 \textbf{ meets } t_2 \wedge t_3 \textbf{ meets } t_4 \Rightarrow t_1 \textbf{ meets } t_4 \vee t_3 \textbf{ meets } t_2 \vee (\exists t'.\ t_1 \textbf{ meets } t' \wedge t' \textbf{ meets } t_4) \vee (\exists t''.\ t_3 \textbf{ meets } t'' \wedge t'' \textbf{ meets } t_2) \quad \text{(A3.4)}$$

$$\neg(t_1 \textbf{ meets } t_2 \wedge t_2 \textbf{ meets } t_1) \quad \text{(A3.5)}$$

$$u \textbf{ sum\_of } (u_1, u_2) =_{def} \forall u'.\ u' \textbf{ overlaps } u \Leftrightarrow (u' \textbf{ overlaps } u_1 \vee u' \textbf{ overlaps } u_2) \quad \text{(D3.1)}$$

We also write u as $u_1 + u_2$ if and only if u **sum_of** (u_1, u_2).

$$u_1 \textbf{ part\_of } (u_1 + u_2) \quad \text{(L3.1)}$$

$$u_2 \textbf{ part\_of } (u_1 + u_2) \quad \text{(L3.2)}$$

$$u_1 \textbf{ part\_of } u \wedge u_2 \textbf{ part\_of } u \Rightarrow (u_1 + u_2) \textbf{ part\_of } u \quad \text{(L3.3)}$$

$$u_1 \textbf{ part\_of } u_2 \Rightarrow (u_1 + u_2) \textbf{ identical\_to } u_2 \quad \text{(L3.4)}$$

$$u_1 \textbf{ part\_of } u_2 \wedge u' \textbf{ part\_of } u'' \wedge \Rightarrow (u_1 + u') \textbf{ part\_of } (u_2 + u'') \quad \text{(L3.5)}$$

$$((u_1 + u_2) + u_3) \textbf{ identical\_to } (u_1 + (u_2 + u_3)) \quad \text{(L3.6)}$$

Proof. L3.1, L3.2 and L3.3 can be proved by D3.1 and A1.5, whereas L3.4 can be proved by L1.15, L3.3, L3.2 and A1.7. To prove L3.5, we assume u_1 **part_of** u_2 and u' **part_of** u'', and prove u_1 **part_of** $(u_2 + u'')$ and u' **part_of** $(u_2 + u'')$ by L1.13 along with L3.1 and L3.2, respectively. We then prove the conclusion by applying L3.3. To prove L3.6, we deduce u_3 **part_of** $(u_2 + u_3)$ and $(u_2 + u_3)$ **part_of** $(u_1 + (u_2 + u_3))$ by L3.2. Therefore u_3 **part_of** $(u_1 + (u_2 + u_3))$ by L1.13. Call this result $\star$. We know u_2 **part_of** $(u_2 + u_3)$ by L3.1 and therefore $(u_1 + u_2)$ **part_of** $(u_1 + (u_2 + u_3))$ by L3.5 and L1.15. This latter result along with the result $\star$ deduced previously tells us that $((u_1 + u_2) + u_3)$ **part_of** $(u_1 + (u_2 + u_3))$ by L3.3. In much the same way we can deduce $(u_1 + (u_2 + u_3))$ **part_of** $((u_1 + u_2) + u_3)$. We then prove the conclusion by applying A1.7. □

$$t \textbf{ concatenation\_of } (t_1, t_2) =_{def} t_1 \textbf{ meets } t_2 \wedge t \textbf{ sum\_of } (t_1, t_2) \quad \text{(D3.2)}$$

$$t_1 \textbf{ starts } t =_{def} \exists t_2.\ t_2 : \mathit{Connected\_Temporal\_Region} \wedge t \textbf{ concatenation\_of } (t_1, t_2) \quad \text{(D3.3)}$$

$$t_2 \textbf{ ends } t =_{def} \exists t_1.\ t_1 : \mathit{Connected\_Temporal\_Region} \wedge t \textbf{ concatenation\_of } (t_1, t_2) \quad \text{(D3.4)}$$

$$t_1 \textbf{ meets } t_2 \Rightarrow \exists t.\ \ t\colon Connected\_Temporal\_Region \wedge t \textbf{ concatenation\_of } (t_1, t_2) \wedge t_1 \textbf{ starts } t \wedge t_2 \textbf{ ends } t \tag{A3.6}$$

$$t \textbf{ concatenation\_of } (t_1, t_2) \Rightarrow duration(t) = duration(t_1) + duration(t_2) \tag{A3.7}$$

$$t_1 \textbf{ earlier\_than } t_2 =_{def} \exists t.\ \ t\colon Connected\_Temporal\_Region \wedge t_1 \textbf{ meets } t \wedge t \textbf{ meets } t_2 \tag{D3.5}$$

$$t_1 \textbf{ earlier\_than\_or\_meets } t_2 =_{def} t_1 \textbf{ earlier\_than } t_2 \vee t_1 \textbf{ meets } t_2 \tag{D3.6}$$

$$t\colon Connected\_Temporal\_Region \Rightarrow \neg(t \textbf{ earlier\_than\_or\_meets } t) \tag{L3.7}$$

$$t_1, t_2\colon Connected\_Temporal\_Region \wedge t_1 \textbf{ earlier\_than\_or\_meets } t_2 \Rightarrow \neg(t_2 \textbf{ earlier\_than\_or\_meets } t_1) \tag{L3.8}$$

Proof. L3.7 follows from A3.5, D3.5 and D3.6. L3.8 follows from A3.4 and L3.7. □

Section 4: Scattered Regions

$$u \textbf{ difference\_of } (u_2, u_1) =_{def} \forall u'.\ \ u' \textbf{ overlaps } u \Leftrightarrow (\exists u''.\ \ u'' \textbf{ part\_of } u_2 \wedge u'' \textbf{ discrete\_from } u_1 \wedge u' \textbf{ overlaps } u'') \tag{D4.1}$$

$$u_1 \textbf{ interior\_part\_of } u_2 =_{def} u_1 \textbf{ proper\_part\_of } u_2 \wedge \exists u.\ \ u \textbf{ difference\_of } (u_2, u_1) \wedge (\forall u'.\ \ u' \textbf{ partially\_overlaps } u_1 \Rightarrow u' \textbf{ overlaps } u) \tag{D4.2}$$

$$u_1 \textbf{ crosses } u_2 =_{def} u_1 \textbf{ overlaps } u_2 \wedge\ \exists u.\ \ u \textbf{ difference\_of } (\mathcal{U}, u_2) \wedge u_1 \textbf{ overlaps } u \tag{D4.3}$$

$$u_1 \textbf{ straddles } u_2 =_{def} \forall u.\ \ u_1 \textbf{ interior\_part\_of } u \Rightarrow u \textbf{ crosses } u_2 \tag{D4.4}$$

$$\neg(u \textbf{ crosses } u) \tag{L4.1}$$

$$u_1 \textbf{ straddles } u_2 \Rightarrow \neg(u_1 \textbf{ interior\_part\_of } u_2) \tag{L4.2}$$

$$u_1 \textbf{ part\_of } u_2 \Rightarrow u_1 \textbf{ interior\_part\_of } u_2 \vee u_1 \textbf{ straddles } u_2 \tag{T4.1}$$

$$u \textbf{ difference\_of } (\mathcal{U}, u_2) \wedge u_1 \textbf{ overlaps } u \Rightarrow \neg(u_1 \textbf{ part\_of } u_2) \tag{L4.3}$$

$$u_1 \textbf{ crosses } u_2 \Rightarrow \neg(u_1 \textbf{ part\_of } u_2) \wedge u_1 \textbf{ overlaps } u_2 \tag{L4.4}$$

$$u_1 \textbf{ partially\_overlaps } u_2 \Rightarrow u_1 \textbf{ crosses } u_2 \wedge u_2 \textbf{ crosses } u_1 \tag{L4.5}$$

Proof. In order to prove L4.1, we assume u **crosses** u. A contradiction is created by unfolding D4.3, D4.1 and D1.5, and by using L1.11. In order to prove L4.2 suppose u_1 **straddles** u_2 and assume u_1 **interior_part_of** u_2. Then u_2 **crosses** u_2 by D4.4 which contradicts L4.1. T4.1 can then be proved by L4.2. We prove L4.3 by contradiction. We know there is some u'' such that u'' **part_of** $\mathcal{U}$ and u'' **discrete_from** u_2 and u_1 **overlaps** u'' by D4.1 and *modus ponens*. If we assume u_1 **part_of** u_2, then for any u''' if u''' **overlaps** u_1 then u''' **overlaps** u_2 by A1.5. Let u''' be u''. Since u'' **overlaps** u_1 by L1.11, we have u'' **overlaps** u_2 which contradicts u'' **discrete_from** u_2 by D1.5. L4.4 can be proved by L4.3 and D4.3. L4.5 follows from D1.11, L4.4 and L1.11. □

$$u' \textbf{ boundary\_of } u =_{def} \forall u''.\; u'' \textbf{ part\_of } u' \Rightarrow u'' \textbf{ straddles } u \tag{D4.5}$$

$$u' \textbf{ closure\_of } u =_{def} \forall u''.\; u'' \textbf{ boundary\_of } u \Rightarrow u' \textbf{ sum\_of } (u, u'') \tag{D4.6}$$

$$u_1 \textbf{ separate\_from } u_2 =_{def} \exists u_1', u_2'.\; u_1' \textbf{ closure\_of } u_1 \wedge u_2' \textbf{ closure\_of } u_2 \Rightarrow u_1' \textbf{ discrete\_from } u_2 \wedge u_1 \textbf{ discrete\_from } u_2' \tag{D4.7}$$

$$u' \textbf{ closure\_of } u \Rightarrow u' \textbf{ identical\_to } u \tag{A4.1}$$

$$u_1 \textbf{ discrete\_from } u_2 \Rightarrow u_1 \textbf{ separate\_from } u_2 \tag{L4.6}$$

Proof. L4.6 can be proved by A4.1 and D4.7 with D1.5 and L1.11. □

$$t', t'' : Connected\_Temporal\_Region \wedge t' \textbf{ interior\_part\_of } t'' \Rightarrow \exists t_1, t_2.\; t_1, t_2 : Connected\_Temporal\_Region \wedge t'' \textbf{ concatenation\_of } (t_1, t', t_2) \tag{A4.2}$$

$$t', t'' : Connected\_Temporal\_Region \wedge t' \textbf{ proper\_part\_of } t'' \wedge t' \textbf{ straddles } t'' \Rightarrow t' \textbf{ starts } t'' \vee t' \textbf{ ends } t'' \tag{A4.3}$$

$$t_1, t_2 : Connected\_Temporal\_Region \wedge t_1 \textbf{ crosses } t_2 \Rightarrow \exists t', t, t''.\; t', t, t'' : Connected\_Temporal\_Region \wedge (t_1 \textbf{ concatenation\_of } (t', t) \wedge t_2 \textbf{ concatenation\_of } (t, t'')) \vee (t_2 \textbf{ concatenation\_of } (t', t) \wedge t_1 \textbf{ concatenation\_of } (t, t'')) \tag{A4.4}$$

$$t' \textbf{ during } t'' =_{def} t' \textbf{ interior\_part\_of } t'' \tag{D4.8}$$

$$t', t'' : Connected\_Temporal\_Region \wedge t' \textbf{ proper\_part\_of } t'' \Rightarrow t' \textbf{ starts } t'' \vee t' \textbf{ ends } t'' \vee t' \textbf{ during } t'' \tag{T4.2}$$

Proof. T4.2 can be proved by T4.1, D4.8 and A4.3. □

$$t_1, t_2 : Connected\_Temporal\_Region \wedge t_1 \textbf{ earlier\_than\_or\_meets } t_2 \Rightarrow t_1 \textbf{ discrete\_from } t_2 \tag{L4.7}$$

Proof. We prove L4.7 by contradicition. Assume $\neg(t_1$ **discrete_from** $t_2)$, *i.e.* t_1 **overlaps** t_2 by D1.5. Then by L1.21 one of the following holds: t_1 **identical_to** t_2 or t_1 **proper_part_of** t_2 or t_2 **proper_part_of** t_1 or t_1 **partially_overlaps** t_2. Consider the case where t_1 **identical_to** t_2. Suppose t_1 **meets** t_2, then a contradiction arises from A3.5. Suppose t_1 **earlier_than** t_2, then there is some t such that t_1 **meets** t and t **meets** t_2 by D3.5, and a contradiction again arises from A3.5. Consider the case where t_1 **proper_part_of** t_2, then t_1 **starts** t_2 or t_1 **ends** t_2 or t_1 **during** t_2 by T4.2. Whether t_1 **meets** t_2 or t_1 **earlier_than** t_2 it is possible to build concatenations using D3.2 with D3.3, D3.4, D4.8 and A4.2 such that a contradiction arises from A3.5. We can similarly create a contradiction for the case where t_2 **proper_part_of** t_1. Now consider the case where t_1 **partially_overlaps** t_2. Then t_1 **crosses** t_2 by L4.5, and there is some t', t and t'' such that either t_1 **concatenation_of** (t', t) and t_2 **concatenation_of** (t, t''), or t_2 **concatenation_of** (t', t) and t_1 **concatenation_of** (t, t'') by A4.4. Whether t_1 **meets** t_2 or t_1 **earlier_than** t_2 it is possible to build concatenations using D3.2 such that a contradiction arises from A3.5. □

$$t_1, t_2 : \mathit{Connected\_Temporal\_Region} \wedge t_1 \textbf{ earlier\_than\_or\_meets } t_2 \Rightarrow t_1 \textbf{ separate\_from } t_2 \quad \text{(T4.3)}$$

Proof. T4.3 can be proved by L4.7 and L4.6. □

Note that in the sequel the formula $\bigwedge_{i=1}^{n-1} x_i \textbf{ rel } x_{i+1}$ can be interpreted as $x_1 \textbf{ rel } x_2 \wedge x_2 \textbf{ rel } x_3 \wedge \ldots \wedge x_{n-1} \textbf{ rel } x_n$ for any relation **rel** between instances $x_1, \ldots, x_n$.

$$\begin{aligned} r : \mathit{Scattered\_Spatial\_Region} =_{def} \exists r_1, \ldots, r_n.\ r_1, \ldots, r_n : \mathit{Connected\_Spatial\_Region} \\ \wedge\, r \textbf{ sum\_of } (r_1, \ldots, r_n) \wedge \bigwedge_{i=1}^{n-1} r_i \textbf{ discrete\_from } r_{i+1} \end{aligned} \quad \text{(D4.9)}$$

$$\begin{aligned} (\mathit{Spatial\_Region} = \mathit{Connected\_Spatial\_Region} \cup \mathit{Scattered\_Spatial\_Region}) \\ \wedge\, (\mathit{Connected\_Spatial\_Region} \cap \mathit{Scattered\_Spatial\_Region} = \emptyset) \end{aligned} \quad \text{(A4.5)}$$

$$r : \mathit{Connected\_Spatial\_Region} \vee r : \mathit{Scattered\_Spatial\_Region} \quad \text{(T4.4)}$$

$$r : \mathit{Scattered\_Spatial\_Region} \Rightarrow r \textbf{ part\_of } \mathcal{R} \quad \text{(L4.8)}$$

$$r \textbf{ part\_of } \mathcal{R} \quad \text{(T4.5)}$$

Proof. T4.4 can be proved by A4.5 and A1.3. L4.8 can be proved by D4.9, L2.11, L3.3 and L3.6. T4.5 can be proved by T4.4, L2.11 and L4.8. □

$$r : \mathit{Spatial\_Region} \Rightarrow \forall t.\ r : \mathit{Spatial\_Region} \textbf{ at } t \quad \text{(A4.6)}$$

$$r_1 \textbf{ part\_of } r_2 \Rightarrow \forall t.\ r_1 \textbf{ part\_of } r_2 \textbf{ at } t \quad \text{(A4.7)}$$

$$\begin{aligned} u : \mathit{Scattered\_Spatiotemporal\_Region} =_{def} \\ \exists u_1, \ldots, u_n.\ u_1, \ldots, u_n : \mathit{Connected\_Spatiotemporal\_Region} \\ \wedge\, u \textbf{ sum\_of } (u_1, \ldots, u_n) \\ \wedge\, \Big(\bigwedge_{i=1}^{n-1} \mathit{time}(u_i) \textbf{ earlier\_than\_or\_meets } \mathit{time}(u_{i+1}) \\ \vee \bigwedge_{i=1}^{n-1} \mathit{space}(u_i) \textbf{ discrete\_from } \mathit{space}(u_{i+1}) \Big) \end{aligned} \quad \text{(D4.10)}$$

$$\begin{aligned} (\mathit{Spatiotemporal\_Region} \\ = \mathit{Connected\_Spatiotemporal\_Region} \cup \mathit{Scattered\_Spatiotemporal\_Region}) \\ \wedge\, (\mathit{Connected\_Spatiotemporal\_Region} \cap \mathit{Scattered\_Spatiotemporal\_Region} = \emptyset) \end{aligned} \quad \text{(A4.8)}$$

$$u : \mathit{Connected\_Spatiotemporal\_Region} \vee u : \mathit{Scattered\_Spatiotemporal\_Region} \quad \text{(T4.6)}$$

$$u : \mathit{Scattered\_Spatiotemporal\_Region} \Rightarrow u \textbf{ part\_of } \mathcal{U} \quad \text{(L4.9)}$$

$$u \textbf{ part\_of } \mathcal{U} \quad \text{(T4.7)}$$

$$u : \mathit{Spatiotemporal\_Region} \wedge \mathcal{U} \textbf{ part\_of } u \Rightarrow u \textbf{ identical\_to } \mathcal{U} \quad \text{(L4.10)}$$

Proof. T4.6 can be proved by A4.8 and A1.3. L4.9 can be proved by D4.10, A2.11, L3.3 and L3.6. T4.7 can be proved by T4.6, A2.11 and L4.9. L4.10 can be proved by T4.7 and A1.7. □

$$t\!:\!\mathit{Scattered\_Temporal\_Region} =_{def} \exists t_1,\ldots,t_n.\ t_1,\ldots,t_n\!:\!\mathit{Connected\_Temporal\_Region} \wedge t\ \textbf{sum\_of}\ (t_1,\ldots,t_n) \wedge \bigwedge_{i=1}^{n-1} t_i\ \textbf{earlier\_than\_or\_meets}\ t_{i+1} \tag{D4.11}$$

$$(\mathit{Temporal\_Region} = \mathit{Connected\_Temporal\_Region} \cup \mathit{Scattered\_Temporal\_Region}) \wedge (\mathit{Connected\_Temporal\_Region} \cap \mathit{Scattered\_Temporal\_Region} = \emptyset) \tag{A4.9}$$

$$t\!:\!\mathit{Connected\_Temporal\_Region} \vee t\!:\!\mathit{Scattered\_Temporal\_Region} \tag{T4.8}$$

$$t\!:\!\mathit{Scattered\_Temporal\_Region} \Rightarrow t\ \textbf{part\_of}\ \mathcal{T} \tag{L4.11}$$

$$t\ \textbf{part\_of}\ \mathcal{T} \tag{T4.9}$$

Proof. T4.8 can be proved by A4.9 and A1.3. L4.11 can be proved by D4.11, L2.7, L3.3 and L3.6, whereas T4.9 can be proved by T4.8, L2.7 and L4.11. □

$$u\!:\!\mathit{Scattered\_Spatiotemporal\_Instant} =_{def} u\!:\!\mathit{Scattered\_Spatiotemporal\_Region} \wedge u\ \textbf{sum\_of}\ (u_1,\ldots,u_n) \wedge u_1\!:\!\mathit{Connected\_Spatiotemporal\_Instant} \wedge \ldots \wedge u_n\!:\!\mathit{Connected\_Spatiotemporal\_Instant} \tag{D4.12}$$

$$u\!:\!\mathit{Scattered\_Spatiotemporal\_Interval} =_{def} u\!:\!\mathit{Scattered\_Spatiotemporal\_Region} \wedge u\ \textbf{sum\_of}\ (u_1,\ldots,u_n) \wedge (u_1\!:\!\mathit{Connected\_Spatiotemporal\_Interval} \vee \ldots \vee u_n\!:\!\mathit{Connected\_Spatiotemporal\_Interval}) \tag{D4.13}$$

$$(\mathit{Spatiotemporal\_Instant} = \mathit{Connected\_Spatiotemporal\_Instant} \cup \mathit{Scattered\_Spatiotemporal\_Instant}) \wedge (\mathit{Connected\_Spatiotemporal\_Instant} \cap \mathit{Scattered\_Spatiotemporal\_Instant} = \emptyset) \tag{A4.10}$$

$$(\mathit{Spatiotemporal\_Interval} = \mathit{Connected\_Spatiotemporal\_Interval} \cup \mathit{Scattered\_Spatiotemporal\_Interval}) \wedge (\mathit{Connected\_Spatiotemporal\_Interval} \cap \mathit{Scattered\_Spatiotemporal\_Interval} = \emptyset) \tag{A4.11}$$

$$\neg(u\ \textbf{part\_of}\ \mathcal{T}) \wedge \neg(u\ \textbf{part\_of}\ \mathcal{R}) \wedge \neg(t\ \textbf{part\_of}\ \mathcal{U}) \wedge \neg(t\ \textbf{part\_of}\ \mathcal{R}) \wedge \neg(r\ \textbf{part\_of}\ \mathcal{U}) \wedge \neg(r\ \textbf{part\_of}\ \mathcal{T}) \tag{A4.12}$$

Section 5: Processual Entities and Independent Continuants

$$\mathit{Processual\_Entity\ is\_a\ Occurrent} \tag{A5.1}$$

$$\mathit{Independent\_Continuant\ is\_a\ Continuant} \tag{A5.2}$$

$$p\ \textbf{has\_participant}\ c\ \textbf{at}\ t \wedge t'\ \textbf{part\_of}\ t \Rightarrow p\ \textbf{has\_participant}\ c\ \textbf{at}\ t' \tag{A5.3}$$

$$P\ \mathit{has\_participant}\ C =_{\mathit{def}} \forall p.\ \ p\!:\!P \Rightarrow \exists c,t.\ \ c\!:\!C\ \textbf{at}\ t \wedge p\ \textbf{has\_participant}\ c\ \textbf{at}\ t \tag{D5.1}$$

$$c\ \textbf{exists\_at}\ t =_{\mathit{def}}\ t\!:\!\mathit{Connected\_Temporal\_Instant} \wedge\ \exists p.\ \ p\ \textbf{has\_participant}\ c\ \textbf{at}\ t \tag{D5.2*}$$

$$p\ \textbf{occurs\_at}\ t =_{\mathit{def}}\ t\!:\!\mathit{Connected\_Temporal\_Instant} \wedge\ \exists c.\ \ p\ \textbf{has\_participant}\ c\ \textbf{at}\ t \tag{D5.3*}$$

$$c\ \textbf{exists\_at}\ t =_{\mathit{def}} \forall t'.\ \ t'\ \textbf{part\_of}\ t \Rightarrow \exists p.\ \ p\ \textbf{has\_participant}\ c\ \textbf{at}\ t' \tag{D5.2}$$

$$p\ \textbf{occurs\_at}\ t =_{\mathit{def}} \forall t'.\ \ t'\ \textbf{part\_of}\ t \Rightarrow \exists c.\ \ p\ \textbf{has\_participant}\ c\ \textbf{at}\ t' \tag{D5.3}$$

$$c\ \textbf{exists\_at}\ t \wedge t'\ \textbf{part\_of}\ t \Rightarrow c\ \textbf{exists\_at}\ t' \tag{A5.4}$$

$$p\ \textbf{occurs\_at}\ t \wedge t'\ \textbf{part\_of}\ t \Rightarrow p\ \textbf{occurs\_at}\ t' \tag{A5.5}$$

$$c\ \textbf{exists\_at}\ t_1 \wedge c\ \textbf{exists\_at}\ t_2 \wedge t\ \textbf{concatenation\_of}\ (t_1,t_2) \Rightarrow c\ \textbf{exists\_at}\ t \tag{A5.6}$$

$$p\ \textbf{occurs\_at}\ t_1 \wedge p\ \textbf{occurs\_at}\ t_2 \wedge t\ \textbf{concatenation\_of}\ (t_1,t_2) \Rightarrow p\ \textbf{occurs\_at}\ t \tag{A5.7}$$

$$\begin{aligned} t\ \textbf{first\_instant\_of}\ p =_{\mathit{def}}\ & t\!:\!\mathit{Connected\_Temporal\_Instant} \wedge p\ \textbf{occurs\_at}\ t \\ & \wedge (\forall t'.\ \ t'\ \textbf{earlier\_than\_or\_meets}\ t \Rightarrow \neg(p\ \textbf{occurs\_at}\ t')) \end{aligned} \tag{D5.4}$$

$$\begin{aligned} t\ \textbf{last\_instant\_of}\ p =_{\mathit{def}}\ & t\!:\!\mathit{Connected\_Temporal\_Instant} \wedge p\ \textbf{occurs\_at}\ t \\ & \wedge (\forall t'.\ \ t\ \textbf{earlier\_than\_or\_meets}\ t' \Rightarrow \neg(p\ \textbf{occurs\_at}\ t')) \end{aligned} \tag{D5.5}$$

$$\begin{aligned} c_1\ \textbf{located\_in}\ c_2\ \textbf{at}\ t =_{\mathit{def}} \exists r_1,r_2.\ \ & c_1\ \textbf{located\_in}\ r_1\ \textbf{at}\ t \wedge c_2\ \textbf{located\_in}\ r_2\ \textbf{at}\ t \\ & \Rightarrow r_1\ \textbf{part\_of}\ r_2 \end{aligned} \tag{D5.6}$$

$$\begin{aligned} C_1\ \mathit{located\_in}\ C_2 =_{\mathit{def}} \forall c_1,t.\ \ c_1\ \textbf{instance\_of}\ C_1\ \textbf{at}\ t \Rightarrow \exists c_2.\ \ & c_2\ \textbf{instance\_of}\ C_2\ \textbf{at}\ t \\ & \wedge c_1\ \textbf{located\_in}\ c_2\ \textbf{at}\ t \end{aligned} \tag{D5.7}$$

$$p\ \textbf{located\_at}\ u \wedge t\ \textbf{difference\_of}\ (\mathcal{T}, \mathit{time}(u)) \Rightarrow p\ \textbf{occurs\_at}\ \mathit{time}(u) \wedge \neg(p\ \textbf{occurs\_at}\ t) \tag{A5.8}$$

$$\begin{aligned} p\ \textbf{located\_at}\ u \wedge p\ \textbf{has\_participant}\ c\ \textbf{at}\ t \wedge c\ \textbf{located\_in}\ r\ \textbf{at}\ t & \\ \Rightarrow r\ \textbf{part\_of}\ \mathit{space}(u) \wedge t\ \textbf{part\_of}\ \mathit{time}(u) & \end{aligned} \tag{A5.9}$$

$$p\!:\mathit{Instantaneously\_Occurring\_Processual\_Entity} \quad \text{(D5.8)}$$
$$=_{def} \exists u, t.\ \ p\ \textbf{located\_at}\ u \wedge t\ \textbf{identical\_to}\ \mathit{time}(u) \wedge t\!:\mathit{Temporal\_Instant}$$

$$p\ \textbf{located\_at}\ u \wedge u\!:\mathit{Connected\_Spatiotemporal\_Region} \wedge t\ \textbf{first\_instant\_of}\ p \quad \text{(A5.10)}$$
$$\Rightarrow t\ \textbf{starts}\ \mathit{time}(u)$$
$$p\ \textbf{located\_at}\ u \wedge u\!:\mathit{Scattered\_Spatiotemporal\_Region} \wedge u\ \textbf{sum\_of}\ (u_1, \ldots, u_n) \quad \text{(A5.11)}$$
$$\wedge\, t\ \textbf{first\_instant\_of}\ p \Rightarrow t\ \textbf{starts}\ \mathit{time}(u_1)$$

$$p'\ \textbf{preceded\_by}\ p =_{def} \exists t, t'.\ \ t, t'\!:\mathit{Connected\_Temporal\_Instant} \quad \text{(D5.9*)}$$
$$\wedge\, p\ \textbf{occurs\_at}\ t \wedge p'\ \textbf{occurs\_at}\ t' \wedge t\ \textbf{earlier\_than}\ t'$$
$$p'\ \textbf{preceded\_by}\ p =_{def} \exists u, u'.\ \ p\ \textbf{located\_at}\ u \wedge p\ \textbf{located\_at}\ u' \quad \text{(D5.9)}$$
$$\wedge\, ((u, u'\!:\mathit{Connected\_Spatiotemporal\_Region} \Rightarrow \mathit{time}(u)\ \textbf{earlier\_than}\ \mathit{time}(u'))$$
$$\vee\, (u\!:\mathit{Connected\_Spatiotemporal\_Region} \wedge u'\!:\mathit{Scattered\_Spatiotemporal\_Region}$$
$$\wedge\, u'\ \textbf{sum\_of}\ (u'_1, \ldots, u'_n) \Rightarrow \mathit{time}(u)\ \textbf{earlier\_than}\ \mathit{time}(u'_1))$$
$$\vee\, (u\!:\mathit{Scattered\_Spatiotemporal\_Region} \wedge u'\!:\mathit{Connected\_Spatiotemporal\_Region}$$
$$\wedge\, u\ \textbf{sum\_of}\ (u_1, \ldots, u_n) \Rightarrow \mathit{time}(u_n)\ \textbf{earlier\_than}\ \mathit{time}(u'))$$
$$\vee\, (u\!:\mathit{Scattered\_Spatiotemporal\_Region} \wedge u'\!:\mathit{Scattered\_Spatiotemporal\_Region}$$
$$\wedge\, u\ \textbf{sum\_of}\ (u_1, \ldots, u_n) \wedge u'\ \textbf{sum\_of}\ (u'_1, \ldots, u'_n)$$
$$\Rightarrow \mathit{time}(u_n)\ \textbf{earlier\_than}\ \mathit{time}(u'_1)))$$

$$t\ \textbf{last\_instant\_of}\ p \wedge t'\ \textbf{first\_instant\_of}\ p' \wedge t\ \textbf{earlier\_than}\ t' \quad \text{(T5.1)}$$
$$\Rightarrow p'\ \textbf{preceded\_by}\ p$$

Proof. T5.1 follows from D5.4, D5.5 and D5.9. □

$$P\ \mathit{preceded\_by}\ P' =_{def} \forall p.\ \ p\!:\!P \Rightarrow \exists p'.\ \ p'\!:\!P \wedge p\ \textbf{preceded\_by}\ p' \quad \text{(D5.10)}$$
$$p'\ \textbf{immediately\_preceded\_by}\ p =_{def} \exists t.\ \ t\ \textbf{last\_instant\_of}\ p \wedge t\ \textbf{first\_instant\_of}\ p' \quad \text{(D5.11*)}$$

$$\begin{aligned}
&p' \textbf{ immediately_preceded_by } p =_{def} &\text{(D5.11)}\\
&\exists u, u'.\ p \textbf{ located_at } u \wedge p \textbf{ located_at } u'\\
&\neg\, (p, p' : \mathit{Instantaneously_Occurring_Processual_Entity})\\
&\wedge\, ((u, u' : \mathit{Connected_Spatiotemporal_Region} \Rightarrow \mathit{time}(u) \textbf{ meets } \mathit{time}(u'))\\
&\vee\, (u : \mathit{Connected_Spatiotemporal_Region} \wedge u' : \mathit{Scattered_Spatiotemporal_Region}\\
&\wedge\, u' \textbf{ sum_of } (u'_1, \ldots, u'_n) \Rightarrow \mathit{time}(u) \textbf{ meets } \mathit{time}(u'_1))\\
&\vee\, (u : \mathit{Scattered_Spatiotemporal_Region} \wedge u' : \mathit{Connected_Spatiotemporal_Region}\\
&\wedge\, u \textbf{ sum_of } (u_1, \ldots, u_n) \Rightarrow \mathit{time}(u_n) \textbf{ meets } \mathit{time}(u'))\\
&\vee\, (u : \mathit{Scattered_Spatiotemporal_Region} \wedge u' : \mathit{Scattered_Spatiotemporal_Region}\\
&\wedge\, u \textbf{ sum_of } (u_1, \ldots, u_n) \wedge u' \textbf{ sum_of } (u'_1, \ldots, u'_n)\\
&\Rightarrow \mathit{time}(u_n) \textbf{ meets } \mathit{time}(u'_1)))
\end{aligned}$$

$$\begin{aligned}
&C' \ \mathit{transformation_of}\ C =_{def} \forall c, t.\ \ c : C \textbf{ at } t \Rightarrow \exists t'.\ \ c : C' \textbf{ at } t' &\text{(D5.12)}\\
&\wedge\, t \textbf{ earlier_than_or_meets } t' \wedge \neg(\exists t''.\ \ c : C \textbf{ at } t'' \wedge c : C' \textbf{ at } t'')
\end{aligned}$$

$$\begin{aligned}
&c' \textbf{ derives_from } c \Rightarrow \forall t.\ \ \neg(c \textbf{ identical_to } c' \textbf{ at } t) &\text{(A.5.12)}\\
&c' \textbf{ derives_from } c \Rightarrow \exists t_1, t_2.\ \ t_1, t_2 : \mathit{Connected_Temporal_Interval} &\text{(A.5.13)}\\
&\wedge\, c \textbf{ exists_at } t_1 \wedge (\forall t'_1.\ \ t_1 \textbf{ earlier_than_or_meets } t'_1 \Rightarrow \neg(c \textbf{ exists_at } t'_1))\\
&\wedge\, c' \textbf{ exists_at } t_2 \wedge (\forall t'_2.\ \ t'_2 \textbf{ earlier_than_or_meets } t_2 \Rightarrow \neg(c' \textbf{ exists_at } t'_2))\\
&\wedge\, \exists t.\ \ t : \mathit{Connected_Temporal_Region} \wedge t \textbf{ ends } t_1 \wedge t \textbf{ starts } t_2\\
&\wedge\, (c \textbf{ located_in } r \textbf{ at } t \wedge c' \textbf{ located_in } r' \textbf{ at } t \Rightarrow r \textbf{ overlaps } r' \textbf{ at } t)
\end{aligned}$$

$$\begin{aligned}
&C' \ \mathit{derives_immediately_from}\ C =_{def} \forall c, t.\ \ c : C \textbf{ at } t &\text{(D5.13)}\\
&\Rightarrow \exists c', t'.\ \ c' : C' \textbf{ at } t' \wedge t \textbf{ earlier_than_or_meets } t' \wedge c' \textbf{ derives_from } c
\end{aligned}$$

$$C_n \ \mathit{derives\_from}\ C_0 =_{def} \bigwedge_{i=0}^{n-1} C_{i+1} \textbf{ derives\_from } C_i \qquad \text{(D5.14)}$$